SL Paper 1

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - x^3$ for which $y = 1$ when $x = 0$. Use Euler's method with a step length of 0.1 to find an approximation for the value of y when $x = 0.4$.

Markscheme

use of $y \to y + h\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

x	y	$\mathrm{d}y/\mathrm{d}x$	$h\mathrm{d}y/\mathrm{d}x$	
0	1	1	0.1	***(A1)***
0.1	1.1	1.33	0.133	***A1***
0.2	1.233	1.866516337	0.1866516337	***A1***
0.3	1.419651634	2.834181181	0.283418118	***A1***
0.4	1.703069752			***(A1)***

Note: After the first line, award ***A1*** for each subsequent y value, provided it is correct to 3sf.

approximate value of $y(0.4) = 1.70$ ***A1***

Note: Accept 1.7 or any answers that round to 1.70.

[7 marks]

Examiners report

[N/A]

Use the integral test to determine whether or not $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

Markscheme

let $u = \ln x$ ***(M1)***

$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \int \frac{1}{u^2}\,\mathrm{d}u$ ***(A1)***

$= -\frac{1}{u} = -\frac{1}{\ln x}$ ***(A1)***

$\int\limits_2^m \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \left[-\frac{1}{\ln x}\right]_2^m$ ***M1***

$= \left[-\frac{1}{\ln m} + -\frac{1}{\ln 2}\right]$ ***A1***

as $m \to \infty, -\frac{1}{\ln m} \to 0$ ***(A1)***

$\int\limits_2^{\infty} \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \frac{1}{\ln 2}$ and hence the series converges ***R1***

[7 marks]

Examiners report

[N/A]

Let

$I_n = \int_1^{\infty} x^n \mathrm{e}^{-x} \mathrm{d}x$ where $n \in \mathbb{N}$.

a. Using l'Hôpital's rule, show that [4]

$\lim\limits_{x \to \infty} x^n \mathrm{e}^{-x} = 0$ where $n \in \mathbb{N}$.

b.i. Show that, for $n \in \mathbb{Z}^+$, [4]

$$I_n = \alpha \mathrm{e}^{-1} + \beta n I_{n-1}$$

where α, β are constants to be determined.

b.ii. Determine the value of I_3, giving your answer as a multiple of e^{-1}. [5]

Markscheme

a. consider $\lim\limits_{x \to \infty} \frac{x^n}{\mathrm{e}^x}$ ***M1***

its value is $\frac{\infty}{\infty}$ so we use l'Hôpital's rule

$\lim\limits_{x \to \infty} \frac{x^n}{\mathrm{e}^x} = \lim\limits_{x \to \infty} \frac{nx^{n-1}}{\mathrm{e}^x}$ ***(A1)***

its value is still $\frac{\infty}{\infty}$ so we need to differentiate numerator and denominator a further $n - 1$ times ***(R1)***

this gives $\lim\limits_{x \to \infty} \frac{n!}{\mathrm{e}^x}$ ***A1***

since the numerator is finite and the denominator $\to \infty$, the limit is zero ***AG***

[4 marks]

b.i. attempt at integration by parts $\left(I_n = -\int_1^{\infty} x^n \mathrm{d}(\mathrm{e}^{-x})\right)$ ***M1***

$I_n = -[x^n \mathrm{e}^{-x}]_1^{\infty} + n\int_1^{\infty} x^{n-1}\mathrm{e}^{-x}\mathrm{d}x$ ***A1A1***

$= \mathrm{e}^{-1} + nI_{n-1}$ ***A1***

$\alpha = \beta = 1$

[9 marks]

b.ii. $I_3 = \mathrm{e}^{-1} + 3I_2$ ***M1***

$= \mathrm{e}^{-1} + 3(\mathrm{e}^{-1} + 2I_1)$ ***A1***

$= 4e^{-1} + 6(e^{-1} + I_0)$ ***A1***

$= 4e^{-1} + 6e^{-1} + 6\int_1^\infty e^{-x}dx$

$= 10e^{-1} - 6[e^{-x}]_1^\infty$ ***A1***

$= 16e^{-1}$ ***A1***

[9 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

Given that y is a function of x, the function z is given by $z = \frac{y-x}{y+x}$, where $x \in \mathbb{R},\ x \neq 3,\ y + x \neq 0$.

a. Show that $\frac{dz}{dx} = \frac{2}{(y+x)^2}\left(x\frac{dy}{dx} - y\right)$. [3]

b. Show that the differential equation $f(x)\left(x\frac{dy}{dx} - y\right) = y^2 - x^2$ can be written as $f(x)\frac{dz}{dx} = 2z$. [2]

c. Hence show that the solution to the differential equation $(x-3)\left(x\frac{dy}{dx} - y\right) = y^2 - x^2$ given that $x = 4$ when $y = 5$ is $\frac{y-x}{y+x} = \left(\frac{x-3}{3}\right)^2$. [7]

Markscheme

a. $z = \frac{y-x}{y+x}$

$\Rightarrow \frac{dz}{dx} = \frac{(y+x)\left(\frac{dy}{dx}-1\right)-(y-x)\left(\frac{dy}{dx}+1\right)}{(y+x)^2}$ ***M1A1***

$\Rightarrow \frac{dz}{dx} = \frac{y\frac{dy}{dx}+x\frac{dy}{dx}-y-x-y\frac{dy}{dx}+x\frac{dy}{dx}-y+x}{(y+x)^2}$ ***A1***

$\Rightarrow \frac{dz}{dx} = \frac{2}{(y+x)^2}\left(x\frac{dy}{dx} - y\right)$ ***AG***

[3 marks]

b. $f(x)\left(\frac{(y+x)^2}{2}\right)\frac{dz}{dx} = y^2 - x^2$ ***(M1)***

$f(x)\frac{dz}{dx} = 2\frac{(y-x)(y+x)}{(y+x)^2}$ ***A1***

$f(x)\frac{dz}{dx} = 2\frac{(y-x)}{(y+x)} = 2z$ ***AG***

[2 marks]

c. **METHOD 1**

$f(x)\frac{dz}{dx} = 2z$

$\frac{1}{z}\frac{dz}{dx} = \frac{2}{f(x)}$

$\frac{1}{z}\frac{dz}{dx} = \frac{2}{x-3}$ ***M1A1***

EITHER

$\Rightarrow \ln z = 2\ln(x-3) + c$ ***A1*** when $y=5,\ x=4 \Rightarrow z=\frac{1}{9}$ ***M1***

$\Rightarrow c=\ln\frac{1}{9}$ ***A1***

$\Rightarrow \ln z=2\ln(x-3)+\ln\frac{1}{9}$

$\Rightarrow \ln z=\ln(x-3)^2-\ln 9$ ***A1***

$\Rightarrow \ln z=\ln\left(\frac{x-3}{3}\right)^2$ ***A1***

$\Rightarrow z=\left(\frac{x-3}{3}\right)^2$

OR

$\Rightarrow \ln z=2\ln(x-3)+\ln c$ ***A1***

$z=c(x-3)^2$ ***M1A1***

when $y=5,\ x=4 \Rightarrow z=\frac{1}{9}$ ***M1***

$\Rightarrow c=\frac{1}{9}$ ***A1***

THEN

$\Rightarrow \frac{y-x}{y+x}=\left(\frac{x-3}{3}\right)^2$ ***AG***

METHOD 2

$f(x)\frac{\mathrm{d}z}{\mathrm{d}x}=2z$

$f(x)\frac{\mathrm{d}z}{\mathrm{d}x}-2z=0$

$\frac{\mathrm{d}z}{\mathrm{d}x}-\frac{2z}{x-3}=0$ ***M1***

integrating factor is $e^{\int\frac{-2}{x-3}\mathrm{d}x}$ ***A1***

$e^{\int\frac{-2}{x-3}\mathrm{d}x}=e^{-2\ln(x-3)}$

$=\frac{1}{(x-3)^2}$ ***A1***

hence $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{z}{(x-3)^2}\right]=0$ ***M1***

$z=A(x-3)^2$ ***A1***

when when $y=5,\ x=4 \Rightarrow z=\frac{1}{9}$ ***M1***

$\Rightarrow A=\frac{1}{9}$ ***A1***

$\Rightarrow \frac{y-x}{y+x}=\left(\frac{x-3}{3}\right)^2$ ***AG***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Find the interval of convergence of the series $\sum_{k=1}^{\infty}\frac{(x-3)^k}{k^2}$.

Markscheme

ratio test $\lim\limits_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim\limits_{k\to\infty}\left|\frac{(x-3)^{k+1}k^2}{(k+1)^2(x-3)^k}\right|$ ***M1A1***

$\lim\limits_{k\to\infty}\left|(x-3)\frac{k^2}{(k+1)^2}\right|$

Note: Condone absence of limits and modulus signs in above.

$|x-3|\lim\limits_{k\to\infty}\left|\left(\frac{1}{1+\frac{1}{k}}\right)^2\right| - |x-3|$ ***A1***

for convergence $|x-3| < 1$ ***(M1)***

$\Rightarrow -1 < x-3 < 1$

$\Rightarrow 2 < x < 4$ ***(A1)***

now we need to test end points. ***(M1)***

when $x = 4$ we have $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$ which is a convergent series ***R1***

when $x = 2$ we have $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ which is convergent ***R1***

(alternating series/absolutely converging series)

hence the interval of convergence is $[2,\ 4]$ ***A1***

Examiners report

This question was well answered in general although the presentation was sometimes poor.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f : x \to \begin{cases} -3x+1 & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ 2x^2-3x+1 & \text{for } x > 0 \end{cases}$.

By considering limits prove that f is

a. continuous at $x = 0$; [4]

b. differentiable at $x = 0$. [5]

Markscheme

a. consider $\lim\limits_{h\to 0+} f(0+h) = \lim\limits_{h\to 0+}(2h^2 - 3h + 1)$ ***M1***

$= 1 = f(0)$ ***A1***

$\lim\limits_{h\to 0-} f(0+h) = \lim\limits_{h\to 0-}(-3h + 1)$ ***M1***

$= 1 = f(0)$ ***A1***

hence f is continuous at $x = 0$ ***AG***

Note: The $= f(0)$ needs only to be seen once.

[4 marks]

b. consider

$\lim_{h \to 0+} \left(\frac{f(0+h)-f(0)}{h} \right) = \lim_{h \to 0+} \left(\frac{2h^2-3h+1-1}{h} \right)$ ***M1A1***

$= \lim_{h \to 0+} \left(\frac{2h^2-3h}{h} \right) = -3$ ***A1***

$\lim_{h \to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0-} \frac{-3h+1-1}{h} = -3$ ***M1A1***

hence f is differentiable at $x = 0$ ***AG***

[5 marks]

Examiners report

a. Again this was a reasonably successful question for many candidates with full marks often being awarded. However a significant minority were let down by giving very informal and descriptive answers which only gained partial marks. As the command term in the question is "prove" there is a need for a degree of formality and an explicit use of limits was expected.

b. Again this was a reasonably successful question for many candidates with full marks often being awarded. However a significant minority were let down by giving very informal and descriptive answers which only gained partial marks. As the command term in the question is "prove" there is a need for a degree of formality and an explicit use of limits was expected.

Consider the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right)$.

a. Show that the series is conditionally convergent but not absolutely convergent. [6]

b. Show that $S > 0.4$. [2]

Markscheme

a. $\sin\left(\frac{1}{n}\right)$ decreases as n increases ***A1***

$\sin\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$ ***A1***

so using the alternating series test, the series is conditionally convergent ***R1***

comparing (the absolute series) with the harmonic series:

$\sum_{n=1}^{\infty} \frac{1}{n}$ ***M1***

$\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$ ***A1***

since the harmonic series is divergent, it follows by the limit comparison theorem that the given series is not absolutely convergent ***R1***

hence the series is conditionally convergent ***AG***

[6 marks]

b. successive partial sums are

$0.841\ldots$

$0.362\ldots$

$0.689\ldots$

$0.441\ldots$ ***A1***

since S lies between any pair of successive partial sums, it follows that S lies between $0.441\ldots$ and $0.689\ldots$ ***R1***

and is therefore greater than 0.4 ***AG***

Note: Use of the facts that the error is always less than the modulus of the next term, or the sequence of even partial sums gives lower bounds are equally acceptable.

[2 marks]

Examiners report

a. [N/A]
b. [N/A]

The function f is defined by

$$f(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x} + 2\cos x}{4},\ x \in \mathbb{R}.$$

The random variable X has a Poisson distribution with mean μ.

a.i. Show that $f^{(4)}x = f(x)$; [4]

a.ii. By considering derivatives of f, determine the first three non-zero terms of the Maclaurin series for $f(x)$. [4]

b.i. Write down a series in terms of μ for the probability $p = \mathrm{P}[X \equiv 0(\bmod 4)]$. [2]

b.ii. Show that $p = \mathrm{e}^{-\mu} f(\mu)$. [1]

b.iii. Determine the numerical value of p when $\mu = 3$. [2]

Markscheme

a.i. $f'(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x} - 2\sin x}{4}$ ***(A1)***

$f''(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x} - 2\cos x}{4}$ ***(A1)***

$f'''(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x} + 2\sin x}{4}$ ***(A1)***

$f^{(4)}(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x} + 2\cos x}{4} = f(x)$ ***AG***

[4 marks]

a.ii. therefore,

$f(0)=1$ and $f^{(4)}(0)=1$ ***(A1)***

$f'(0)=f''(0)=f'''(0)=0$ ***(A1)***

the sequence of derivatives repeats itself so the next non-zero derivative is $f^{(8)}(0)=1$ ***(A1)***

the MacLaurin series is $1+\frac{x^4}{4!}+\frac{x^8}{8!}(+\ldots)$ ***(M1)A1***

[4 marks]

b.i. $p=\mathrm{P}(X=0)+\mathrm{P}(X=4)+\mathrm{P}(X=8)+\ldots$ ***(M1)***

$=\frac{\mathrm{e}^{-\mu}\mu^0}{0!}+\frac{\mathrm{e}^{-\mu}\mu^4}{4!}+\frac{\mathrm{e}^{-\mu}\mu^8}{8!}+\ldots$ ***A1***

[??? marks]

b.ii. $p=\mathrm{e}^{-\mu}\left(1+\frac{\mu^4}{4!}+\frac{\mu^8}{8!}+\ldots\right)$ ***A1***

$=\mathrm{e}^{-\mu}f(\mu)$ ***AG***

[??? marks]

b.iii. $p=\mathrm{e}^{-3}\left(\frac{\mathrm{e}^3+\mathrm{e}^{-3}+2\cos 3}{4}\right)$ ***(M1)***

$=0.226$ ***A1***

[??? marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]

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a. Find the general solution of the differential equation $(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}=1+xy$, for $|x|<1$. [7]

b. (i) Show that the solution $y=f(x)$ that satisfies the condition $f(0)=\frac{\pi}{2}$ is $f(x)=\frac{\arcsin x+\frac{\pi}{2}}{\sqrt{1-x^2}}$. [6]

(ii) Find $\lim\limits_{x\to -1} f(x)$.

Markscheme

a. rewrite in linear form ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{-x}{1-x^2}\right)y=\frac{1}{1-x^2}$

attempt to find integrating factor ***M1***

$I=e^{\int\frac{-x}{1-x^2}\mathrm{d}x}=e^{\frac{1}{2}\ln(1-x^2)}$ ***A1***

$=\sqrt{1-x^2}$ ***A1***

multiply by I and attempt to integrate ***(M1)***

$y(1-x^2)^{\frac{1}{2}}=\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ ***(A1)***

$y(1-x^2)^{\frac{1}{2}} = \arcsin x + c$ ***A1***

[7 marks]

b. (i) attempt to find ***c*** ***M1***

$\frac{\pi}{2} = 0 + c$ ***A1***

so $f(x) = \frac{\arcsin x + \frac{\pi}{2}}{\sqrt{1-x^2}}$ ***AG***

(ii) $\lim\limits_{x \to -1} f(x) = \frac{0}{0}$, so attempt l'Hôpital's rule ***(M1)***

consider $\lim\limits_{x \to -1} \frac{\frac{1}{(1-x^2)^{\frac{1}{2}}}}{\frac{-x}{(1-x^2)^{\frac{1}{2}}}}$ ***A1A1***

$= 1$ ***A1***

[6 marks]

Examiners report

a. A good number of wholly correct answers were seen to this question and for stronger candidates it proved to be a successful question. A small number of candidates failed to recognise part (a) as needing an integrating factor. More commonly, students left out the negative sign in the integrating factor or were unable to simplify the integrating factor.

b. A good number of wholly correct answers were seen to this question and for stronger candidates it proved to be a successful question. In part (b) many students recognised the use of L'Hopital's rule, but in a number of cases made errors in the differentiation.

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a. Calculate the following limit [3]

$\lim\limits_{x \to 0} \frac{2^x - 1}{x}$.

b. Calculate the following limit [5]

$\lim\limits_{x \to 0} \frac{(1+x^2)^{\frac{3}{2}} - 1}{\ln(1+x) - x}$.

Markscheme

a. $\lim\limits_{x \to 0} \frac{2^x - 1}{x} = \lim\limits_{x \to 0} \frac{2^x \ln 2}{1}$ ***M1A1***

$= \ln 2$ ***A1***

[3 marks]

b. **EITHER**

$\lim\limits_{x \to 0} \frac{(1+x^2)^{\frac{3}{2}} - 1}{\ln(1+x) - x} = \lim\limits_{x \to 0} \frac{3x(1+x^2)^{\frac{1}{2}}}{\frac{1}{1+x} - 1}$ ***M1A1***

$= \lim_{x \to 0} \frac{3x(1+x^2)^{\frac{1}{2}}(1+x)}{-x}$ ***A1A1***

$= -3$ ***A1***

OR

$(1 + x^2)^{\frac{3}{2}} - 1 = 1 + \frac{3}{2}x^2 + \ldots - 1 = \frac{3}{2}x^2 + \ldots$ ***M1A1***

$\ln(1 + x) - x = x - \frac{1}{2}x^2 + \ldots - x = -\frac{1}{2}x^2 + \ldots$ ***M1A1***

Limit $= -3$ ***A1***

[5 marks]

Examiners report

a. [N/A]
b. [N/A]

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a. By evaluating successive derivatives at $x = 0$, find the Maclaurin series for $\ln \cos x$ up to and including the term in x^4 . [8]

b. Consider $\lim_{x \to 0} \frac{\ln \cos x}{x^n}$, where $n \in \mathbb{R}$. [5]

Using your result from (a), determine the set of values of n for which

(i) the limit does not exist;

(ii) the limit is zero;

(iii) the limit is finite and non-zero, giving its value in this case.

Markscheme

a. attempt at repeated differentiation (at least 2) ***M1***

let $f(x) = \ln \cos x$, $f(0) = 0$ ***A1***

$f'(x) = -\tan x$, $f'(0) = 0$ ***A1***

$f''(x) = -\sec^2 x$, $f''(0) = -1$ ***A1***

$f'''(x) = -2\sec^2 x \tan x$, $f'''(0) = 0$ ***A1***

$f^{iv}(x) = -2sec^4 x - 4sec^2 x tan^2 x$, $f^{iv}(0) = -2$ ***A1***

the Maclaurin series is

$\ln \cos x = -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$ ***M1A1***

Note: Allow follow-through on final ***A1***.

[8 marks]

b. $\frac{\ln \cos x}{x^n} = -\frac{x^{2-n}}{2} - \frac{x^{4-n}}{12} + \ldots$ ***(M1)***

(i) the limit does not exist if $n > 2$ ***A1***

(ii) the limit is zero if $n < 2$ ***A1***

(iii) if $n = 2$, the limit is $-\frac{1}{2}$ ***A1A1***

[5 marks]

Examiners report

a. A reasonable number of candidates achieved full marks on this question. However, in part (a) a number of candidates struggled to find the Maclaurin series accurately. It was not uncommon to see errors in finding the higher derivatives, which was often caused by not simplifying the answer for earlier derivatives. At this level, it is expected that candidates understand the importance of using the most efficient methods.

b. A pleasing number of candidates made significant progress or achieved full marks in part (b), provided that they realised the importance of recognizing that

$$\frac{\ln \cos x}{x^n} = -\frac{x^{2-n}}{2} - \frac{x^{4-n}}{12} + \ldots$$

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a. Given that the series $\sum_{n=1}^{\infty} u_n$ is convergent, where $u_n > 0$, show that the series $\sum_{n=1}^{\infty} u_n^2$ is also convergent. [4]

b.i. State the converse proposition. [1]

b.ii. By giving a suitable example, show that it is false. [1]

Markscheme

a. since $\sum u_n$ is convergent, it follows that $\lim_{n \to \infty} u_n = 0$ ***R1***

therefore, there exists N such that for $n \geqslant N,\ u_n < 1$ ***R1***

Note: Accept as n gets larger, eventually $u_n < 1$.

therefore (for $n \geqslant N$), $u_n^2 < u_n$ ***R1***

by the comparison test, $\sum u_n^2$ is convergent ***R1***

[4 marks]

b.i. the converse proposition is that if $\sum u_n^2$ is convergent, then $\sum u_n$ is also convergent ***A1***

[1 mark]

b.ii. a suitable counter-example is

$u_n = \frac{1}{n}$ $\left(\text{for which } \sum u_n^2 \text{ is convergent but } \sum u_n \text{ is not convergent}\right)$ ***A1***

[1 mark]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

b. (i) Sum the series $\sum_{r=0}^{\infty} x^r$. [11]

(ii) **Hence**, using sigma notation, deduce a series for

(a) $\frac{1}{1+x^2}$;

(b) $\arctan x$;

(c) $\frac{\pi}{6}$.

c. Show that $\sum_{n=1}^{100} n! \equiv 3(\bmod 15)$. [4]

Markscheme

b. (i) $\sum_{r=0}^{\infty} x^r = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$ ***A1***

(ii) (a) replacing ***x*** by $-x^2$ gives ***(M1)***

$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \ldots$ ***A1***

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$ ***(A1)***

$= \sum_{r=0}^{\infty} (-1)^r x^{2r}$ ***A1*** ***N2***

(b) $\arctan x = \int \frac{\mathrm{d}x}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + c$ ***M1A1***

$x = 0 \Rightarrow c = 0$ ***A1***

$\arctan x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{2r+1}$ ***A1***

(c) by taking $x = \frac{1}{\sqrt{3}}$ ***M1***

$\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6} = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{\sqrt{3}}\right)^{2r+1}}{2r+1}$ ***A1***

[11 marks]

c. $\sum_{n=1}^{100} n! = 1! + 2! + 3! + 4! + 5! + \ldots$ ***M1*** □□□□□

$= 1 + 2 + 6 + 24 + 120 + \ldots$

$\equiv 1 + 2 + 6 + 24 + 0 + 0 + 0 + \ldots (\bmod 15)$ ***M1A1*** □□

$\equiv 33(\bmod 15)$ ***A1***

$\equiv 3(\bmod 15)$ ***AG***

[4 marks]

Examiners report

b. In part (b) many did not recognize the sum of a simple geometric series to infinity and got involved in some heavy Maclaurin work thus wasting time. The clear instruction "Hence" was ignored by many candidates so that the question became more difficult and time consuming than it should have been.

c. Part (c) proved not to be as difficult as expected.

(a) Assuming the Maclaurin series for e^x, determine the first three non-zero terms in the Maclaurin expansion of $\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$.

(b) The random variable X has a Poisson distribution with mean μ. Show that $\mathrm{P}\left(X \equiv 1(\bmod 2)\right) = a + b\mathrm{e}^{c\mu}$ where a, b and c are constants whose values are to be found.

Markscheme

(a) $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

$\mathrm{e}^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$ ***A1***

$\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ ***(M1)A1***

Note: Accept any valid (otherwise) method.

[3 marks]

(b) $\mathrm{P}\left(X \equiv 1(\bmod 2)\right) = \mathrm{P}(X = 1,\ 3,\ 5,\ \ldots)$ ***(M1)***

$= \mathrm{e}^{-\mu}\left(\mu + \frac{\mu^3}{3!} + \frac{\mu^5}{5!} + \ldots\right)$ ***A1***

$= \frac{\mathrm{e}^{-\mu}(\mathrm{e}^{\mu} - \mathrm{e}^{-\mu})}{2}$ ***A1***

$= \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2\mu}$ ***A1***

$\left(a = \frac{1}{2},\ b = -\frac{1}{2},\ c = -2\right)$

[4 marks]

Examiners report

[N/A]

Use l'Hôpital's rule to find $\lim\limits_{x \to 0}(\csc x - \cot x)$.

Markscheme

$\lim\limits_{x \to 0}(\csc x - \cot x) = \lim\limits_{x \to 0}\left(\frac{1 - \cos x}{\sin x}\right)$ ***M1A1***

$= \lim\limits_{x \to 0}\left(\frac{\sin x}{\cos x}\right)$ ***M1A1***

$= 0$ ***A1***

Examiners report

This question was well answered in general although some of the weaker candidates differentiated the whole expression rather than the numerator and denominator separately. Some candidates wrote $\cos\mathrm{ec}x - \cot x = \frac{1}{\sin x} - \frac{1}{\tan x} = \frac{\tan x - \sin x}{\sin x \tan x}$ which is correct but it introduced an extra round of differentiation with opportunity for error.

. QUESTION 1 [[N/A

. QUESTION 1 [[N/A

. Using l'Hôpital's Rule, determine the value of [6]

$$\lim_{x \to 0} \frac{\tan x - x}{1 - \cos x}.$$

Markscheme

. MARKSCHEME 1

. MARKSCHEME 1

. $\lim\limits_{x \to 0} \frac{\tan x - x}{1 - \cos x} = \lim\limits_{x \to 0} \frac{\sec^2 x - 1}{\sin x}$ ***M1A1A1***

this still gives $\frac{0}{0}$

EITHER

repeat the process ***M1***

$= \lim\limits_{x \to 0} \frac{2\sec^2 x \tan x}{\cos x}$ ***A1***

$= 0$ ***A1***

OR

$= \lim\limits_{x \to 0} \frac{\tan^2 x}{\sin x}$ ***M1***

$= \lim\limits_{x \to 0} \frac{\sin x}{\cos^2 x}$ ***A1***

$= 0$ ***A1***

[6 marks]

Examiners report

. [N/A]
. [N/A]
. [N/A]

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a. (i) Find the range of values of n for which $\int_1^\infty x^n \mathrm{d}x$ exists. [7]

(ii) Write down the value of $\int_1^\infty x^n \mathrm{d}x$ in terms of n , when it does exist.

b. Find the solution to the differential equation [8]

$$(\cos x - \sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\cos x + \sin x)y = \cos x + \sin x\ ,$$

given that $y = -1$ when $x = \frac{\pi}{2}$.

Markscheme

a. (i) $\int_1^b x^n \mathrm{d}x = \left[\frac{x^{n+1}}{n+1}\right]_1^b, n \neq -1$ ***M1***

$= \frac{b^{n+1}}{n+1} - \frac{1}{n+1}$ ***A1***

$\int_1^b x^n \mathrm{d}x = [\ln x]_1^b = \ln b$ when $n = -1$ ***A1***

if $n + 1 > 0$, $\lim\limits_{b\to\infty}\left[\frac{b^{n+1}}{n+1} - \frac{1}{n+1}\right]$ does not exist since b^{n+1} increases without limit ***R1***

if $n + 1 < 0$, $\lim\limits_{b\to\infty}\left[\frac{b^{n+1}}{n+1} - \frac{1}{n+1}\right]$ exists since $b^{n+1} \to 0$ as $b \to \infty$ ***R1***

if $n = -1$, $\lim\limits_{b\to\infty}[\ln b]$ does not exist since $\ln b$ increases without limit ***R1***

(so integral exists when $n < -1$)

(ii) $\int_1^b x^n \mathrm{d}x = \frac{1}{n+1}$, $(n < -1)$ ***A1***

[7 marks]

b. $(\cos x - \sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\cos x + \sin x)y = \cos x + \sin x$

$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\cos x + \sin x}{\cos x - \sin x}y = \frac{\cos x + \sin x}{\cos x - \sin x}$ ***M1***

IF $\mathrm{e}^{\int \frac{\cos x + \sin x}{\cos x - \sin x}\mathrm{d}x} = \mathrm{e}^{-\ln(\cos x - \sin x)} = \frac{1}{\cos x - \sin x}$ ***M1A1A1***

$\frac{y}{\cos x - \sin x} = \int \frac{\cos x + \sin x}{(\cos x - \sin x)^2}\mathrm{d}x$ ***(M1)***

$\frac{1}{\cos x - \sin x} + k$ ***A1***

Note: Award the above ***A1*** even if k is missing.

$y = 1 + k(\cos x - \sin x)$

$x = \frac{\pi}{2}$, $y = -1$

$-1 = 1 + k(-1)$ ***M1***

$k = 2$

$y = 1 + 2(\cos x - \sin x)$ ***A1***

Note: It is acceptable to solve the equation using separation of variables.

[8 marks]

Examiners report

a. This was found to be the most difficult question on the paper. Whilst the question looked straightforward, indiscriminate use of the infinity symbol showed a lack of appreciation of the subtleties involved in the question. Many of the refinements required in each section were not considered. Knowledge of improper integrals was very poor. Perhaps integration is seen too easily as the application of a number of rules without much thought.

b. Generally this was quite well done. However, many candidates did not realize that it could be solved using an integrating factor and used substitution or variables separable. While these last two approaches could work, the algebra involved soon became unmanageable. This led to many mistakes.

The function f is defined by $f(x) = \mathrm{e}^x \cos x$.

a. Show that $f''(x) = -2\mathrm{e}^x \sin x$. [2]

b. Determine the Maclaurin series for $f(x)$ up to and including the term in x^4 . [5]

c. By differentiating your series, determine the Maclaurin series for $\mathrm{e}^x \sin x$ up to the term in x^3 . [4]

Markscheme

a. $f'(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$ ***A1***

$f''(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \mathrm{e}^x \sin x - \mathrm{e}^x \cos x$ ***A1***

$= -2\mathrm{e}^x \sin x$ ***AG***

[2 marks]

b. $f'''(x) = -2\mathrm{e}^x \sin x - 2\mathrm{e}^x \cos x$ ***A1***

$f^{IV}(x) = -4\mathrm{e}^x \cos x$ ***A1***

$f(0) = 1,\ f'(0) = 1\ ,f''(0) = 0,\ f'''(0) = -2,\ f^{IV}(0) = -4$ ***(A1)***

the Maclaurin series is

$\mathrm{e}^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \dots$ ***M1A1***

Note: Accept multiplication of series method.

[5 marks]

c. differentiating,

$\mathrm{e}^x \cos x - \mathrm{e}^x \sin x = 1 - x^2 - \frac{2x^3}{3} + \dots$ ***M1A1***

$\mathrm{e}^x \sin x = 1 + x - \frac{x^3}{3} + \dots - 1 + x^2 + \frac{2x^3}{3} + \dots$ ***M1***

$= x + x^2 + \frac{x^3}{3} + \dots$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

---

a. Differentiate the expression $x^2 \tan y$ with respect to x, where y is a function of x. [3]

b. Hence solve the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + x \sin 2y = x^3 \cos^2 y$ given that $y = 0$ when $x = 1$. Give your answer in the form $y = f(x)$. [7]

Markscheme

a. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \tan y) = 2x \tan y + x^2 \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1***

b. $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 2x \sin y \cos y = x^3 \cos^2 y$ ***A1***

$\Rightarrow x^2 \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} + 2x \tan y = x^3$ ***M1A1***

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^2 \tan y) = x^3$ ***A1***

$x^2 \tan y = \frac{x^4}{4} + c$ ***A1***

Note: Condone the omission of c in the line above

when $x = 1,\ y = 0 \Rightarrow c = -\frac{1}{4}$ ***M1***

$\tan y = \frac{x^2}{4} - \frac{1}{4x^2}$

$y = \arctan\left(\frac{x^2}{4} - \frac{1}{4x^2}\right)$ ***A1***

Examiners report

a. Many candidates made the connection between (a) and (b) and went on to solve the differential equation correctly. Candidates who failed to make the connection usually tried to write the equation in the form required for the use of an integrating factor but this led nowhere.

b. Many candidates made the connection between (a) and (b) and went on to solve the differential equation correctly. Candidates who failed to make the connection usually tried to write the equation in the form required for the use of an integrating factor but this led nowhere.

Solve the differential equation $x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sqrt{1 + x^2}$

given that $y = 1$ when $x = \sqrt{3}$.

Markscheme

Rewrite the equation in the form

$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x} y = \frac{\sqrt{1+x^2}}{x}$ ***M1A1***

Integrating factor $= e^{\int \left(\frac{2}{x}\right) dx}$ ***M1***

$= e^{2\ln x}$ ***(A1)***

$= x^2$ ***A1***

The equation becomes

$\frac{d}{dy}(yx^2) = x\sqrt{1+x^2}$ ***M1***

$yx^2 = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$ ***A1***

$3 = \frac{1}{3} \times 8 + C \rightarrow C = \frac{1}{3}$ ***M1A1***

$\left[\text{giving } yx^2 = \frac{1}{3}\left((1+x^2)^{\frac{3}{2}} + 1\right)\right]$

[9 marks]

Examiners report

[N/A]

Consider the infinite series $S = \sum_{n=1}^{\infty} \frac{x^n}{2^{2n}(2n^2-1)}$.

(a) Determine the radius of convergence.

(b) Determine the interval of convergence.

Markscheme

(a) let T_n denote the nth term

consider

$\frac{T_{n+1}}{T_n} = \frac{x^{(n+1)}}{2^{2(n+1)}\left(2[n+1]^2-1\right)} \times \frac{2^2(2n^2-1)}{x^n}$ ***M1***

$= \frac{x}{2^2} \times \frac{(2n^2-1)}{\left(2[n+1]^2-1\right)}$ ***A1***

$\rightarrow \frac{x}{4}$ as $n \rightarrow \infty$ ***A1***

so the radius of convergence is 4 ***A1***

[4 marks]

(b) we need to consider $x = \pm 4$ ***R1***

$S(4) = \sum_{n=1}^{\infty} \frac{1}{(2n^2-1)}$ ***A1***

$S(4) < \sum_{n=1}^{\infty} \frac{1}{n^2}$ ***M1***

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent; therefore by the comparison test $S(4)$ is convergent ***R1***

$S(-4) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n^2-1)}$ ***A1***

EITHER

this series is convergent because it is absolutely convergent ***R1***

OR

this series is alternating and is convergent ***R1***

THEN

the interval of convergence is therefore $[-4, 4]$ ***A1***

Note: The final ***A1*** is independent of any of the previous marks.

[7 marks]

Examiners report

[N/A]

Solve the following differential equation

$$(x+1)(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x + 1$$

giving your answer in the form $y = f(x)$.

Markscheme

Rewrite the equation in the form

$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{(x+1)(x+2)} = \frac{1}{x+2}$ ***M1***

Integrating factor $= \exp\left(\int \frac{\mathrm{d}x}{(x+1)(x+2)}\right)$ ***A1***

$= \exp\left(\int\left(\frac{1}{x+1} - \frac{1}{x+2}\right)\mathrm{d}x\right)$ ***M1A1***

$= \exp\ln\left(\frac{x+1}{x+2}\right)$ ***A1***

$= \frac{x+1}{x+2}$ ***A1***

Multiplying by the integrating factor,

$\left(\frac{x+1}{x+2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{(x+2)^2} = \frac{x+1}{(x+2)^2}$ ***M1***

$= \frac{x+2}{(x+2)^2} - \frac{1}{(x+2)^2}$ ***A1***

Integrating,

$\left(\frac{x+1}{x+2}\right)y = \ln(x+2) + \frac{1}{x+2} + C$ ***A1A1***

$y = \left(\frac{x+2}{x+1}\right)\left\{\ln(x+2) + \frac{1}{x+2} + C\right\}$ ***A1***

[11 marks]

Examiners report

[N/A]

Given that $\frac{\mathrm{d}x}{\mathrm{d}y} + 2y\tan x = \sin x$, and $y = 0$ when $x = \frac{\pi}{3}$, find the maximum value of y.

Markscheme

integrating factor $= \mathrm{e}^{\int 2\tan x dx}$ ***M1***

$= \mathrm{e}^{2\ln\sec x}$ ***A1***

$= \sec^2 x$ ***A1***

it follows that

$y\sec^2 x = \int \sin x \sec^2 x \mathrm{d}x$ ***M1***

$= \int \sec x \tan x \mathrm{d}x$ ***(A1)***

$= \sec x + C$ ***A1***

substituting,

$0 = 2 + C$ so $C = -2$ ***M1A1***

the solution is

$y = \cos x - 2\cos^2 x$ ***A1***

EITHER

using a GDC

maximum value of y is 0.125 ***A2***

OR

$y' = -\sin x + 4\sin x\cos x = 0$ ***M1***

$\Rightarrow \cos x = \frac{1}{4}$ (or $\sin x = 0$ which leads to a minimum)

$\Rightarrow y = \frac{1}{8}$ ***A1***

[11 marks]

Examiners report

Solutions were often disappointing with some candidates even unable to find the integrating factor because of an inability to integrate $2\tan x$. Some candidates who found the integrating factor correctly were then unable to integrate $\sin x\sec^2 x$ and others omitted the constant of integration. Some of the candidates who obtained the correct expression for ***y*** failed to realise that the quickest way to find the maximum value was to plot the graph of \(y|) on their calculator.